

# Infra-red Abelian dominance without Abelian-projection

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## Abstract

Maximal Abelian gauge has been a particular choice to study dynamical generation of off-diagonal gluon masses in QCD. This gauge is a special case of Abelian projection. Massive off-diagonal gluons are considered a clue to Abelian dominance. Here we propose a gauge condition which is quadratic in fields and which does not fall in the class of Abelian projection. However it does generate off-diagonal gluon masses dynamically thus hinting at Abelian dominance in this gauge too.

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## I. INTRODUCTION

One of the most burning questions to be answered definitively in QCD is, what is the physical mechanism by which quarks and gluons are confined. Classically confinement is expressed as linear inter-quark potential. In mid 70s, a dual version of type-II superconductor was proposed by Nambu[1], 't Hooft[2] and Mandelstam[3]. In type-II superconductor magnetic field is trapped in the form of one dimensional Abrikosov vortex tubes inside the superconductor i.e., inside the medium of condensed electric charges[4]. In the same way but dually the idea of these proposals for quark confinement mechanism is that the electric field due to quarks is trapped in the form of vortex tubes in a phase in which magnetic monopoles are condensed so that the confinement forces can be characterised by constant string tension or as a linear inter-quark potential. Here the key point is that the dual picture is based on Abelian gauge theory whereas QCD is non-Abelian, therefore one needs to demonstrate that QCD reduces to an effective Abelian theory at infra-red scale. Secondly it requires a new concept of condensed magnetic monopoles. The need for such an effective theory led to the concept of “Abelian dominance”[5].

According to Abelian dominance, at low energy scale, QCD can be effectively expressed in terms of Abelian gauge degrees of freedom [5]. It is usually discussed in terms of off-diagonal gluons i.e., gluons not associated with Cartan subalgebra of  $SU(N)$ . In the  $SU(N)$  gauge theory there are  $N(N - 1)$  off-diagonal gluons. These gluons attaining a large dynamical mass is presumed to provide the required evidence of existence of Abelian dominance. At infra-red energies below this dynamical mass scale these off-diagonal components would become inactive and decouple, and only dominant degrees are now diagonal gluons since they remain massless. Thus one gets  $N - 1$  copies of Abelian gauge theory at low energy scale corresponding to each diagonal gluon. As far as we know the occurrence of off-diagonal gluon masses and infra-red Abelian dominance have been studied mostly in maximal Abelian gauge, a few of the references being [6–9], which is a particular case of Abelian projection[2].

An Abelian projection[2] is a partial gauge fixing which leaves the maximal torus group of a group  $G$  unbroken. For  $SU(N)$ , the gauge condition takes the form of variable  $X(x)$  satisfying following conditions:

- It takes values in lie algebra of  $SU(N)$ .

- It transforms according to adjoint action

$$X(x) \rightarrow U(x)X(x)U(x)^{-1} \quad (1)$$

We can perform gauge rotation and diagonalize  $X(x)$

$$\widehat{X(x)} = V(x)X(x)V(x)^{-1}; \quad V(x) \in SU(N) \quad (2)$$

This diagonalized  $X(x)$  is invariant under  $U(1)^{N-1}$ , maximal torus group of  $SU(N)$ . Hence each such variable  $X(x)$  defines an Abelian projection. We propose “a quadratic gauge” which doesn’t fall in the class of Abelian projection and which results in a dynamical mass generation for off-diagonal gluons, which in turn leads to Abelian dominance. So to the best of our knowledge this paper presents one substantial ingredient of the confinement without Abelian projection.

## II. A QUADRATIC GAUGE AND EFFECTIVE LAGRANGIAN

An effective Lagrangian of the theory is obtained through gauge fixing. Unlike the usual gauges here we choose a different gauge condition, namely constraining the operator  $A^{\mu a}A_\mu^a$  viz., “ a quadratic gauge ”. i.e,

$$F^a[A^\mu(x)] = A_\mu^a(x)A^{\mu a}(x) = f^a(x); \text{ for each } a \quad (3)$$

where  $f^a(x)$  is an arbitrary function of  $x$ . This clearly is not an Abelian projection as the gauge condition doesn’t take values in the Lie algebra. Furthermore, the condition of abelian projection is stipulated to be covariant in order to ensure survival of an abelian component. The quadratic gauge does not meet this condition either. This gauge condition results in the gauge fixing and ghost contributions to the effective Lagrangian of the form

$$\mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghost}} = -\frac{1}{2\zeta} \sum_a (A_\mu^a A^{\mu a})^2 - \sum_a \bar{c}^a A^{\mu a} (D_\mu c)^a \quad (4)$$

Now onwards we shall drop  $\sum$  but summation over  $a$  will be understood where it appears repeatedly, including when repeated *thrice* as in the ghost terms above. In particular,

$$\mathcal{L}_{\text{ghost}} = -\bar{c}^a A^{\mu a} (D_\mu c)^a = -\bar{c}^a A^{\mu a} \partial_\mu c^a + g f^{abc} \bar{c}^a c^c A^{\mu a} A_\mu^b \quad (5)$$

where the summation over indices  $a, b$  and  $c$  each runs independently over 1 to  $N^2 - 1$ . With this understanding we write the effective Lagrangian as

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{2\zeta}(A_\mu^a A^{\mu a})^2 - \overline{c^a} A^{\mu a} (D_\mu c)^a \quad (6)$$

where first term is Yang-Mills Lagrangian with  $F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) - gf^{abc}A_\mu^b(x)A_\nu^c(x)$ .

### III. OFF-DIAGONAL GLUON MASS GENERATION AND ABELIAN DOMINANCE IN A QUADRATIC GAUGE

Although we do not intend to derive perturbative rules for the  $S$ -matrix here, the intuitive understanding in terms of the properties of quanta that can in principle occur in the asymptotic states is the most convenient in taking the discussion forward. As such we now proceed to identify propagators which allows us to make the hypothesis of ghost condensation. Then we proceed to deriving mass terms of the effective degrees of freedom using the ghost condensation hypothesis.

#### A. Mass generation due to the ghost condensation

Gluon propagator is obtained from Yang-Mills Lagrangian in the standard form,

$$\mathcal{O}_{\mu\nu}^{-1ab}(p) = -\frac{\delta^{ab}}{p^2} \left( \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad (7)$$

On the other hand the ghosts do not possess any propagator in the theory since there is no free quadratic part of ghost fields in the effective Lagrangian in eq. (6). Hence for the propagator of a ghost field we have

$$G^{ab}(p) = 0 \quad (8)$$

However the ghost term in eq. (6) does contain ghost-gluon interaction terms. Thus while the ghosts are non-propagating spectators, they continue to interact through the gluons. Consider therefore, the individual terms in the ghost Lagrangian  $\mathcal{L}_{\text{ghost}}$

$$-\overline{c^a} A^{\mu a} (D_\mu c)^a = -\overline{c^a} A^{\mu a} \partial_\mu c^a + gf^{abc} \overline{c^a} c^c A^{\mu a} A_\mu^b \quad (9)$$

where the summation over the indices goes as explained in sec. II. We see that if the ghosts undergo condensation, the gluons acquire a mass matrix. Such a possibility was elaborated

in Ref. [6]. We shall here show, that a similar condensation mechanism is responsible for the generation of off-diagonal gluon mass in the quadratic gauge. We assume that ghost fields acquire non-zero real ghost-anti-ghost condensation i.e.,

$$\langle \bar{c}^i c^j \rangle \neq 0 \quad \text{for all } i \text{ and } j. \quad (10)$$

This may be understood as a consequence of the interaction among the spectator ghost fields mediated by the gluons. In the proposed ghost condensed phase, second term of eq. (9) gives us off-diagonal components of gluon mass matrix

$$(M^2)_{\text{dyn}}^{ab} = 2g \sum_{c=1}^{N^2-1} f^{abc} \langle \bar{c}^a c^c \rangle \quad (11)$$

whereas diagonal components of  $M_{\text{dyn}}^2$  are zero since  $f^{aac} = 0$ . To obtain a spectrum of the theory i.e., to obtain masses of gluons, we must diagonalize the matrix and find eigenvalues.

The required demonstration is simple in an  $SU(N)$  symmetric state, where ghost-anti-ghost condensates are taken to be identical i.e.,

$$\langle \bar{c}^1 c^1 \rangle = \dots = \langle \bar{c}^1 c^{N^2-1} \rangle = \dots = \langle \bar{c}^{N^2-1} c^1 \rangle = \dots = \langle \bar{c}^{N^2-1} c^{N^2-1} \rangle = K \quad (12)$$

Thus

$$(M^2)_{\text{dyn}}^{ab} = 2g \sum_{c=1}^{N^2-1} f^{abc} K \quad (13)$$

Hence now this gluon mass matrix is just an antisymmetric matrix formed by structure constants of  $SU(N)$ ,

$$J^{ab} = \left[ \sum_{c=1}^{N^2-1} f^{abc} \right] \quad (14)$$

where  $a$  is a row index and  $b$  is a column index. What we claim is, this matrix has exactly  $N(N-1)$  non-zero eigenvalues and thus nullity is exactly  $N-1$ .

Mathematics of this result is the following: structure constants themselves furnish  $(N^2-1) \times (N^2-1)$  adjoint representation. So,  $J^{ab}$  is equal to sum over generators in adjoint representation i.e.,  $i \left( \sum_{c=1}^{N^2-1} [T^{ab}]^c \right) \in \mathfrak{g}, \text{ lie algebra vector space}$ . Coadjoint representation  $[T_{ab}]_c^*$  which is dual of the adjoint is the same as the adjoint for  $SU(N)$ . Elements  $g$  of  $SU(N)$  acts on  $\mathfrak{g}^*, \text{ dual vector space}$  by conjugation

$$\{Ad^* F = g^{-1} F g, \quad F \in \mathfrak{g}^*\}$$

The orbit  $\mathcal{O}_F = \{Ad^*F, \forall g \in SU(N)\}$ , passing through  $F$ , is known as coadjoint orbit. Now, any point  $F \in \mathfrak{g}^*$  in compact connected lie group has maximal torus group as a stabilizer. Hence for  $SU(N)$  maximal torus group  $U(1)^{N-1}$  is also a stabilizer. So the coadjoint orbits of  $SU(N)$  group are isomorphic to  $SU(N)/U(1)^{N-1}$  i.e.,  $\mathcal{O}_F \sim SU(N)/U(1)^{N-1} \sim \mathbb{C}P^{N-1} \otimes \mathbb{C}P^{N-2} \otimes \dots \otimes \mathbb{C}P^1$ . So, it is  $N(N-1)$  dimensional symplectic manifold with symplectic form  $Tr(F.[T^a, T^b])$ , where  $F \in \mathfrak{g}^*$  and  $T^a, T^b \in \mathfrak{g}$ , defined on it.[10]. In the present problem, the given matrix  $J^{ab} = \frac{-i}{N}Tr(F.[T^a, T^b])$  with  $F = \sum_{c=1}^{N^2-1} T^c$  and  $T^a, T^b, T^c$  being basis generators. The rank of a symplectic form is always equal to the dimension of a coadjoint orbit. Since given matrix is normal, its rank and number non-zero eigenvalues are equal. So the rank and therefore the number of non-zero eigenvalues are  $N(N-1)$ . Hence nullity is  $N-1$ . Thus we have proved that in an  $SU(N)$  invariant vacuum,  $N(N-1)$  off-diagonal gluons acquire masses and  $N-1$  diagonal gluons remain massless.

The non-zero eigenvalues thus identified, being eigenvalues of antisymmetric matrix, are pure imaginary and occur in conjugate pairs. viz.,  $M_{gluon}^2 = \pm im^2$  ( $m^2$  positive real). This implies, mass of these gluons  $M_{gluon} = \frac{1}{\sqrt{2}}(1 \pm i)m$  or  $\frac{1}{\sqrt{2}}(-1 \mp i)m$ . We ignore the latter one since it gives  $Re(M_{gluon})$  negative which is not physical. Because we are not interested in an  $S$ -matrix interpretation for these degrees of freedom, *prima facie* there is no danger from these eigenvalues being pure imaginary. However to retain the intuitive appeal of the arguments it is necessary to check that we have not departed too far from their interpretation as quanta and in principle an  $S$ -matrix interpretation. This is what we shall do in the next section.

We now emphasize the implication of the fact that the off-diagonal gluons do have real part to their mass. Since they are massive they can not mediate long range interaction beyond the range of inverse of its mass,  $r \lesssim M^{-1}$ . Therefore below mass scale of lightest off-diagonal gluon  $M_{gluon}^{lig}$ , no off-diagonal gluons will contribute to any interaction. So, diagonal gluons are the only degrees of freedom that will mediate long range interactions as they are massless. This way one ends up getting an effective description of theory with  $N-1$  copies of Abelian gauge theory. Thus if this description as commonly used in the literature can be relied on, then we have proved existence of Abelian dominance at infra-red scale in this gauge.

#### IV. HERMITICITY OF THE EFFECTIVE LAGRANGIAN IN A QUADRATIC GAUGE

In order to retain the appeal to a picture of this ground state in terms of quanta it is useful to check that it will not conflict with nominally expected restrictions on the manner which such degrees of freedom enter into an  $S$ -matrix. Here we will show that while in the normal phase the Lagrangian is manifestly hermitian, a condition indispensable for  $S$ -matrix unitarity[11], the effective Lagrangian obeys an extended hermiticity condition also in the ghost phase,

The effective Lagrangian in the normal phase is given in eq. (6)

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{2\zeta}(A_\mu^a A^{\mu a})^2 - \bar{c}^a A^{\mu a} (D_\mu c)^a \quad (15)$$

The hermiticity property of fields is given by

$$\begin{aligned} A_\mu^{a\dagger} &= A_\mu^a \\ c^{a\dagger} &= c^a \\ \bar{c}^{a\dagger} &= -\bar{c}^a \end{aligned} \quad (16)$$

It is easy to check that this Lagrangian is hermitian under hermitian conjugation of fields since

$$\begin{aligned} (\bar{c}^a c^c)^\dagger &= -c^c \bar{c}^a = \bar{c}^a c^c \quad \text{and} \\ (\bar{c}^a A^{\mu a} \partial_\mu c^a)^\dagger &= -\partial_\mu c^a A^{\mu a} \bar{c}^a = \bar{c}^a A^{\mu a} \partial_\mu c^a \end{aligned}$$

(we have used anti commutativity of ghost fields)

In the ghost condensed phase, the effective Lagrangian now becomes

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{2\zeta}(A_\mu^a A^{\mu a})^2 - \bar{c}^a A^{\mu a} (\partial_\mu c)^a + M_a^2 A_\mu^a A^{\mu a} \quad (17)$$

Here  $M_a^2 = 0$  when  $a$  indexes the diagonal gluons, e.g, for  $SU(3)$ ,  $M_3^2 = M_8^2 = 0$ . While for the off-diagonal gluons,  $M_1^2 = +im_1^2, M_2^2 = -im_1^2, M_4^2 = +im_2^2, M_5^2 = -im_2^2, M_6^2 = +im_3^2, M_7^2 = -im_3^2$  ( $m_1^2, m_2^2, m_3^2$  are positive real), hence for  $SU(3)$  last term of the effective Lagrangian in eq. (17) would be

$$\begin{aligned} M_a^2 A_\mu^a A^{\mu a} &= +im_1^2 A_\mu^1 A^{\mu 1} - im_1^2 A_\mu^2 A^{\mu 2} + im_2^2 A_\mu^4 A^{\mu 4} - im_2^2 A_\mu^5 A^{\mu 5} \\ &\quad + im_3^2 A_\mu^6 A^{\mu 6} - im_3^2 A_\mu^7 A^{\mu 7} \end{aligned} \quad (18)$$

Now taking hermitian conjugate of the Lagrangian in eq. (17) would alter nothing in first two terms and hermiticity of the third has been proven above but, it will interchange the sign of mass term between “conjugate” gluons. e.g, in eq. (18)

$$\begin{aligned} (M_a^2 A_\mu^a A^{\mu a})^\dagger = & -im_1^2 A_\mu^1 A^{\mu 1} + im_1^2 A_\mu^2 A^{\mu 2} - im_2^2 A_\mu^4 A^{\mu 4} + im_2^2 A_\mu^5 A^{\mu 5} \\ & - im_3^2 A_\mu^6 A^{\mu 6} + im_3^2 A_\mu^7 A^{\mu 7} \end{aligned} \quad (19)$$

We now invoke triality[12], a property of all known hadrons, which keeps them color singlet. In the condensed state we require any observable excitation above this state to possess zero triality. Define the triality operator  $\mathfrak{T}$  as

$$\begin{aligned} \mathfrak{T}H_1\mathfrak{T}^\dagger &= H_2 & \mathfrak{T}H_4\mathfrak{T}^\dagger &= H_5 & \mathfrak{T}H_6\mathfrak{T}^\dagger &= H_7 & \mathfrak{T}H_3\mathfrak{T}^\dagger &= H_8 \\ \mathfrak{T}H_2\mathfrak{T}^\dagger &= H_1 & \mathfrak{T}H_5\mathfrak{T}^\dagger &= H_4 & \mathfrak{T}H_7\mathfrak{T}^\dagger &= H_6 & \mathfrak{T}H_8\mathfrak{T}^\dagger &= H_3 \end{aligned} \quad (20)$$

Where  $H_i$  can be  $-\frac{1}{4}F_{\mu\nu}^i F^{\mu\nu i}$ ,  $-\frac{1}{2\zeta}(A_\mu^i A^{\mu i})^2$ ,  $-\overline{c}^i A^{\mu i}(\partial_\mu c)^i$ ,  $im^2 A_\mu^i A^{\mu i}$ . It can be seen that the triality operation does not cause any change in first three terms in eq. (17) and further when operated on eq. (19) it can be easily verified that

$$\mathfrak{T}(M_a^2 A_\mu^a A^{\mu a})^\dagger \mathfrak{T}^\dagger = M_a^2 A_\mu^a A^{\mu a} \quad (21)$$

Thus the vacuum may be considered to furnish a non-trivial representation of triality. So as a result  $\mathcal{L}_{\text{eff}}$  in eq. (17) is invariant under hermitian conjugation followed by triality

$$\mathfrak{T}\mathcal{L}_{\text{eff}}^\dagger \mathfrak{T}^\dagger = \mathcal{L}_{\text{eff}} \quad (22)$$

Which replaces usual unitarity condition for  $S$ - matrix,  $S^\dagger S = SS^\dagger = 1$  by  $\mathfrak{T}S^\dagger \mathfrak{T}^\dagger S = S\mathfrak{T}S^\dagger \mathfrak{T}^\dagger = 1$  in the ghost condensed phase. We may understand the inclusion of triality symmetry in ensuring the unitarity of the  $S$ -matrix to be a refinement over the usual discrete internal symmetry  $C$ , the charge conjugation symmetry.

## V. CONCLUSION

We have proposed the set of gauge conditions  $A_\mu^a A^{\mu a} = f^a(x)$ , for each  $a$ , and shown that in a vacuum characterised by ghost-anti-ghost condensation which respects  $SU(N)$  invariance, the off-diagonal gluons acquire masses dynamically. This result strongly supports the existence of Abelian dominance at infra-red energies in quadratic gauge in QCD. We also



proved that in the quadratic gauge, the effective Lagrangian in the absence of condensation is manifestly hermitian, giving us usual unitarity condition for the  $S$ -matrix whereas in the ghost condensed phase it satisfies extended hermiticity condition, giving a modified unitarity condition for the  $S$ -matrix .

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